**Chap 2: Laplace Transform**

1. **The Laplace Transform**
2. **Laplace Transform**

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| **Definition & Notation** | * The Laplace transform of a function is:   + is the complex variable. The domain of is called frequency domain   + is the real variable. The domain of is called time domain   + is the kernel of the transformation   + is the Laplace Transform Operator   + is called the Laplace transform pair * Notation: * The region of convergence |
| **Existence of Laplace Transform** | * A function is said to be of exponential order if there exist positive constants and such that: * If is continuous and of exponential order ⇒ then exists for |
| Example | Find the exponential of ?   * We have: (because ) * So ⇒ is of exponential order |

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| **Improper Integral** | * The Laplace Transform can be found using the improper integral: |
| **Laplace Transform Table** | * The table of usually used Laplace Transform  |  |  |  | | --- | --- | --- | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  | |  |  |  |  * The properties of Laplace Transform:  |  |  | | --- | --- | |  | | | Linearity |  | | First shift theorem |  | | Derivative of transform |  | |
| Example | * We have: * Apply the property of Derivative of transform: * We have: * Apply the property of Derivative of transform: |

1. **Inverse Laplace Transform**

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| **Inverse Laplace Transform** | * If is the Laplace transform of , then is called the inverse Laplace transform of :   + is the inverse Laplace transform operator |
| Example | * We have: |

1. **Solving Differential Equations Using Laplace Transform**

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| **Transform of Derivative** | * 1st order derivative: | * 2nd order derivative: |
| * **General case:** | |
| **Transform of Integral** | * **General case:** | |

1. **Solving Differential Equation**

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| * Apply the Laplace Transform: * Apply the transform of derivative: * Taking the inverse Laplace Transform we have: |
| * Apply the Laplace Transform: * Apply the transform of derivative: * Taking the inverse Laplace Transform we have: |

1. **Solving Simultaneous Differential Equation**

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| * Apply the Laplace Transform: * Apply the transform of derivative: * Find first: * Taking the inverse Laplace Transform we have: * Find – Cách 1: * Find – Cách 2: |

1. **Step & Impulse Functions**
2. **Heaviside Step function**

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| **Heaviside Step function** | * The Heaviside step function: * The unit step function: |  | | |
| * The top hat function: | | |  |
| * The piecewise-continuous function: | |  | |
| * Express the piecewise-continuous function using the **unit step function**: * Express the piecewise-continuous function using the **top hat function**: | | | |
| Example | Express using unit step function?  Express using top hat function? | | | |

1. **Laplace Transform of unit step function**

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| **Laplace Transform of unit step function** | * **General case:** * Special case : |
| **Second Shift Theorem** | * If , then with : |
| Example | Let . Find ?   * We express using the unit step function: * Apply the Laplace transform of unit step function:   Let . Find ?   * We express using the unit step function: * We must express as the function of : * Apply thesecond shift theorem: |

1. **Inversion using second shift theorem**

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| **Inversion using second shift theorem** | * If , then with : |
| Example | * Let: * First we find the inverse Laplace transform of : * Therefore, we have: * Let: * First we find the inverse Laplace transform of : * Therefore, we have: |
| **Solving Differential Equation** | * We express using the unit step function: * Using Laplace transform, we have: * Apply Laplace transform we have: * Taking the inverse Laplace Transform we have: |

1. **Periodic Function**

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| **Periodic Function** | * A function is a periodic functions with period if for all integer : | |
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| **Laplace Transform of periodic function** | * General formula:   **Use the Heaviside step function to find the Laplace Transform of periodic function**   * Express the periodic function using the top hat function: * Find the Laplace Transform for : | |
| Example | is a periodic function. Find ?   * First, we have: which means * We have: * Apply the Laplace Transform: | |

1. **Impulse Function**
2. **Impulse Function**

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| **Impulse Function** | * Impulse functions are functions that only have a very large value over a very short interval * The impulse function whose magnitude is unity is called the **unit impulse function** or **Dirac delta function** or **delta function** * The unit impulse occurring at is denoted by and having the following properties: * The unit impulse function at time is denoted by and having the following properties: |  |
| **Sifting Theorem** | * If is continuous at then | |
| **Laplace Transform of Impulse Function** | * General Formula for Laplace Transform:  |  |  | | --- | --- | |  |  | |  |  | |  |  | |  |  | |  |  | | |

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| Example | * We have: * We have: * So: |
| **Solving Differential Equation** | * Apply the Laplace Transform: * Taking the inverse Laplace Transform we have: |

1. **Relationship between Heaviside step & Impulse functions**

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| **Relationship between Heaviside step &**  **Impulse functions** | * We have: * The **generalized derivatives** of piecewise-continuous functions having jump discontinuities d1, d2,..., dn at times t1, t2,…, tn   Where | |
| Example | . Find the generalized derivative? | |
| * We have: |  |
| * Therefore, we have: | |
|  | Where |
| * We have: * Therefore: * Apply the Laplace Transform: * Taking the inverse Laplace Transform we have: | |

1. **Transfer Function**

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| **Definition** | **Definition:**  The **transfer function** of a linear time-invariant system under the assumption that **all the initial conditions are 0** (the system is **in a quiescent state**)  **How to find the Transfer Function?**   * If we have a linear time-invariant system characterized by the differential equation: * Then we obtain the Laplace Transform with **all the initial conditions are 0**: * Then we obtain the **Transfer Function**   **Characteristic:**   * The transfer function may be expressed as * is the **characteristic equation** of the system   + Its order determines the **order** of the system   + Its roots are referred to as the **poles** of the transfer function * The root of are referred to as the **zeros** of the transfer function | |
| Example | 1. Determine the transfer function characterizing the system 2. Write down the characteristic equation of the system. What is the order of the system? 3. Determine the transfer function poles and zeros, and illustrate them diagrammatically in the s plane 4. Assuming all the initial conditions to be 0, taking Laplace transforms: 5. The characteristic equation of the system:  * The order of the system is 2  1. The transfer function poles are the roots of the characteristic equation | |
| * The transfer function zeros are the roots of |  |
| **Stability** | * If we have an equation is the characteristic equation of a causal time-invariant linear system   + **Stability**: when all the **poles** of thetransfer function **have negative real parts** (in the left half of the s plane)   + **Unstability**: when there is a **pole** of thetransfer function **doesn’t have a negative real part**   + **Marginally Stability**: when there is **a complex pole that doesn’t have the real part** * **Routh-Hurwitz criterion**: All roots of an equation have **negative real parts** when the determinants **Δ1, Δ2,... are all positive**, where: | |
| Example | . Check stability?   * The poles of the system are:   . Check stability?   * The poles of the system are:   . Check stability?   * The poles of the system are:   . Check stability?   * The poles of the system are:   . Check stability?   * The poles of the system are: | |
| . Show that the roots of the characteristic equation all have negative real parts?   * We have: * So:       Thus Δ1 > 0, Δ2 > 0, Δ3 > 0 and Δ4 > 0, so that all the roots of the given characteristic equation have negative real parts | |
| **Impulse Response** | * We have: * Let the input then the system response will be determined by * The impulse response of the system (weighting function) is: | |
| Example | . Find the impulse response of the system?   * Let and all initial conditions are 0 * Taking the Laplace Transform: * The impulse response of the system:   . Find   * The Laplace Transform of : * The Laplace Transform of : * The Transform Function of the system is: | |
| **Initial- and final-value theorems** | * **Initial-Value Theorem**: If and are both Laplace-transformable and if exists then: * **Final-Value Theorem**: If and are both Laplace-transformable and if exists then: * **Steady-state gain (SSG)** or Steady-state errors:   With | |
| Example | . Find the initial & final value?   * Use the initial-value theorem: * Use the final-value theorem:   . Find the steady-state gain of a system?   * We have * Therefore: * The steady-state gain of a system: | |

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| **Convolution** | * Given 2 piecewise-continuous functions and , the convolution of and , denoted by , is defined as: * is called “convolution integral”, “superposition integral”, “Duhamel integral”, “folding integral” and “faltung integral” * Note: |
| **Convolution theorem for Laplace transforms** | * If and are of exponential order , piecewise-continuous on and have Laplace transforms and respectively, then, for * The inverse Laplace Transform: |
| Example | * We have: * Apply the convolution theorem for Laplace transforms: * We have: * Apply the convolution theorem for Laplace transforms: * Using the standard method we have: |